Partial Order: $\forall x, y, z \in A$: Reflexive: $x\mathcal{R}x$, Anti-symmetric: $x\mathcal{R}y, y\mathcal{R}x \implies x = y$, Transitive: $x\mathcal{R}y, y\mathcal{R}z \implies x\mathcal{R}z$.
Total Order: $\forall x, y \in A, x\mathcal{R}y \vee y\mathcal{R}x$
Equivalence Relation: $\forall x, y, z \in A$: Reflexive: $x\mathcal{R}x$, Symmetric: $x\mathcal{R}y = y\mathcal{R}x$, Transitive: $x\mathcal{R}y, y\mathcal{R}z \implies x\mathcal{R}z$.
Equivalence Class: $[x] := \{y \in A : x \sim y\}$
Ordered Fields: A field with a partial order ($\leq$) s.t.: ***(i)*** If $x, y, z \in \mathbb{F},\ x < y \implies x + z < x + y$, ***(ii)*** $x, y \in \mathbb{F},\ x, y > 0 \implies xy > 0$
Rational Zeros Theorem: *Suppose* $c_0, \cdots, c_n \in \mathbb{Z},\ r \in \mathbb{Q}$ *satisfies* $c_n r^n + \cdots + c_1 r + c_0 = 0$ *for some* $n \in \mathbb{N},\ c_n \neq 0$. *Let* $r = \frac{c}{d}, c, d \in \mathbb{Z}, d \neq 0$, *be coprime. Then* c, d *divides* c_0, c_n.
LUBP: Given $A \subseteq \mathbb{E}$ where $\mathbb{E}$ is an ordered set, $\exists \sup A \in \mathbb{E} \iff A \neq \emptyset, A \subseteq \mathbb{E}, A$ is bounded above. $\sup A := \alpha,\ \exists \alpha, \beta \in \mathbb{E}$ s.t. $\forall a \in A,\ a \leq \alpha \leq \beta$.
GLBP: Given $A \subseteq \mathbb{E}$ where $\mathbb{E}$ is an ordered set, $\exists \inf A \in \mathbb{E} \iff A \neq \emptyset, A \subseteq \mathbb{E}, A$ is bounded below. $\inf A := \alpha,\ \exists \alpha, \beta \in \mathbb{E}$ s.t. $\forall a \in A,\ \beta \leq \alpha \leq a$.
Archemedian Property: If $y \in \mathbb{R},\ x > 0$, then $\exists n \in \mathbb{N}$ s.t. $n \cdot x > y$. Put $x = 1 : \exists n \in \mathbb{N}$ s.t. $n > y$. Put $y = 1 : \exists n \in \mathbb{N}$ s.t. $n \cdot x > 1 \rightsquigarrow 0 < \frac{1}{n} < x$.
Density of $\mathbb{Q}$ in $\mathbb{R}$: $\forall x, y \in \mathbb{R} : x < y, \exists p \in \mathbb{Q} : x < p < y$
Sequence: A function $f : \mathbb{N} \to \mathbb{R} \iff n \mapsto f(n) \iff n \mapsto f_n$ e.g. $(1, \frac{1}{2}, \frac{1}{3}, \cdots),\ x_n = \frac{1}{n}\ \forall n \in \mathbb{N},\ \{x_n : n \in \mathbb{N}\},\ (x_n)_{n=1}^{\infty},\ (x_n)_{n \in \mathbb{N}}$
Convergent: A sequence (x_n) converges to $x \in \mathbb{R}$ if: $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N,\ |x_n - x| < \varepsilon$. We write $(x_n) \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n := x$, where x is the **limit** of (x_n). **Divergent:** A sequence that does not **converge**.
Triangle Inequality: $|x + y| \leq |x| + |y| \implies |x - y| = |x + (-z + z) - y| \leq |x - z| + |z - y|\ \forall x, y, z \in \mathbb{R}$.
Unique Limits: $x_n \to x,\ x_n \to y \implies x = y$. $|x - y| = |x + (-x + x) - y| \leq |x_n - x| + |x_n - y| = \varepsilon$ if $|x_n - x|, |x_n - y| \leq \frac{\varepsilon}{2}$.
Algebraic Limit Theorem: $x_n \to x, y_n \to y \implies$ ***(i)*** $ax_n \to ax$, ***(ii)*** $x_n \pm y_n \to x \pm y$, ***(iii)*** $x_n \cdot y_n \to x \cdot y$ ***(iv)*** $\frac{x_n}{y_n} \to \frac{x}{y}, y \neq 0$
Monotone Convergence Theorem: Monotone inc/dec and bounded above/below $\implies (x_n)$ converges.
Bolzono-Weirstrauss Theorem: Bounded $\implies \exists (x_{n_k})$ that converges.
Squeeze Theorem: Given $(x_n),\ (y_n),\ (z_n) : y_n \leq x_n \leq z_n \forall n \in \mathbb{N}$ and $y_n \to x,\ z_n \to x$ as $n \to \infty$, $x_n \to x$ as $n \to \infty$.
Test for Divergence: $(x_n) \not\to 0 \implies \sum x_n$ does not converge.
Cauchy Sequence: $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n, m > N, |x_n - x_m| < \varepsilon$. ***Note:*** (x_n) is cauchy $\iff (x_n)$ converges in $\mathbb{R}$ only.
Geometric Series: Given $x \in \mathbb{R}$, $S_n = \sum\limits_{k=1}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$ if $x \neq 1$. $|x| < 1 \implies S_n \to \frac{1}{1-x} \implies (x)^n \to 0$ by ALT. $|x| > 1 \implies S_n \to +\infty$.
Comparison Test: Assume $y_n \geq 0\ \forall n \geq N$. If $|x_n| \leq y_n \forall n \in \mathbb{N}$, then:
(i) $\sum y_n$ converges $\implies \sum x_n$ converges.
(ii) $\sum |x_n|$ diverges $\implies \sum y_n$ diverges.
(iii) $\sum y_n \to +\infty$ & $x_n \geq y_n, \forall n \in \mathbb{N} \implies \sum x_n \to +\infty$.
Absolute Convergence Test: $\sum |x_n|$ converges $\implies \sum x_n$ converges.
Cauchy Condensation Test: Given (x_n) decreasing and nonnegative, $\sum\limits_{n=1}^{\infty} x_n$ converges $\iff \sum\limits_{n=1}^{\infty} 2^n x_{2^n}$ converges.
Cauchy Criterion: $\sum\limits_{n=1}^{\infty} x_n$ converges $\iff \forall \varepsilon > 0, \exists N \in \mathbb{N} : n > m \geq N \implies |x_{m+1} + \cdots + x_n| < \varepsilon$.
p-series Test: $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff p > 1$.
Ratio Test: Given $x_n \neq 0,\ \lim\limits_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| = L$ converges absolutely if $L < 1$, diverges if $L > 1$, inconclusive if $L = 1$.
Root Test: Given $x_n,\ \lim\limits_{n\to\infty} |x_n|^{\frac{1}{n}} = L$ converges absolutely if $L < 1$, diverges if $L > 1$, inconclusive if $L = 1$.
Alternating Series Test: If a sequence (x_n) is decreasing and converges to 0, then $\sum (-1)^{n+1} x_n$ converges.
Exponent rules with e: $x^a = e^{a \log x}$. **Log growth:** $\log n \leq n^a \forall a \in \mathbb{R}^+$. **Diff of Cubes:** $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$.
Closed Set: A set A that contains all of its limit points L_A. **Compact Set:** Closed and bounded. **Open Set:** Not closed.
Limit Points: $A \subseteq \mathbb{R}$. $\exists (x_n) \subseteq A : x_n \neq c\ \forall n \in \mathbb{N} \wedge \lim\limits_{n\to\infty} x_n = c \implies c \in L_A$.
Functional Limit: $A \subseteq R,\ c \in L_A,\ f : A \to R$, $\text{dom}(f) = A$. Then, $\lim\limits_{x\to c} f(x) = L$ if $\forall (x_n) \subseteq A,\ x_n \neq c,\ x_n \to c,\ \lim\limits_{n\to\infty} f(x_n) = L$.
Functional Limit (ε, δ): $\lim\limits_{x\to c} f(x) = L \iff$ for $c \in L_A$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. whenever $x \in A, 0 < |x - c| < \delta$, we have $f(x) - L| < \varepsilon$.
Existance of Limits: $\lim\limits_{x\to c} f(x)$ exists $\iff \lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x)$
Divergence F.L.: If $\exists (x_n), (y_n) \subseteq A : x_n \neq c,\ y_n \neq c\ \forall n$ and $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = c$ and $\lim\limits_{n\to\infty} f(x_n) \neq \lim\limits_{n\to\infty} f(y_n)$, then $\lim\limits_{x\to c} f(x)$ DNE.
Quantitative F.L.: $\lim\limits_{x\to c} f(x) = L \iff \forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon, c) > 0 : 0 < |x - c| < \delta\ (x \in A) \implies |f(x) - L| < \varepsilon$.
Continuity (ε, δ): $f : A \to \mathbb{R}$ is continuous at $c \in A$ if $\forall \varepsilon > 0, \exists \delta > 0 : x \in A, |x - c| < \delta \implies f(x) - f(c)| < \varepsilon$.
C/L: $c \in L_A \implies \Big[f$ cts at $c \iff \lim\limits_{x\to c} f(x) = f(c) \Big]$.
Heine-Borel Theorem: $K \subseteq \mathbb{R}$ compact $\iff K$ is closed and bounded.
Cts Theorem: $f : A \to \mathbb{R}$ cts on A. $K \subseteq A$ compact $\implies f(K)$ is compact. (i.e. f is bounded ($\exists M > 0 : \forall x \in K,\ |f(x)| \leq M$)).
EVT: $f : K \to \mathbb{R}$ cts and K compact $\implies \exists x_0, x_1 \in K : f(x_0) \leq f(x) \leq f(x_1)\ \forall x \in K$.
IVT: $f : [a, b] \to \mathbb{R}$ cts, $L \in \mathbb{R} : f(a) < L < f(b)$ (or $f(b) < L < f(a)$) $\implies \exists c \in (a, b) : f(c) = L$.
Uni Cts: $\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0 : |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. uni cts on $A \implies$ cts on A, cts on compact $K \implies$ uni cts.
Uni Cts: f uni cts $A \iff \forall \varepsilon > 0,\ \exists \delta > 0 : \sup\limits_{\substack{x,y \in A \\ |x-y|<\delta}} |f(x) - f(y)| < \varepsilon \iff \sup \{|f(x) - f(y)| : x, y \in A, |x - y| < \delta\}$.
Non-Uni Cts: f not uni cts $\iff \exists \varepsilon_0 > 0 \wedge (x_n), (y_n) : |x_n - y_n| \to 0) \wedge |f(x_n) - f(y_n)| \geq \varepsilon_0$.
Derivative: $\exists \lim \in \mathbb{R} \implies f'(c) := \lim\limits_{x\to c} \frac{f(x) - f(c)}{x - c}$. **Chain Rule:** $(g \circ f)'(c) := f'(c) g'(f(c))$.
Differentiability: $\lim\limits_{x\to c^-} \frac{f(x) - f(c)}{x - c} = \lim\limits_{x\to c^+} \frac{f(x) - f(c)}{x - c}$
Linear Approximation: $\lim\limits_{x\to c} \frac{f(x) - f(c)}{x - c}$ exists $\iff \exists L, R \in \mathbb{R} : \lim\limits_{x\to c} R(x) = 0$ and $f(x) = f(c) + (x - c)L + (x - c)R(x)$.
Interior EVT (Derivatives): $c \in I$ is an extremum for f and f is diff at $c \implies f'(c) = 0$.
Location of Extrema: $f : [a, b] \to \mathbb{R}$ cts on $[a, b]$, diff on $(a, b) \implies f$ has extrema at either: $a \vee b \vee c \in (a, b) : f'(c) = 0$.
MVT: $f : [a, b] \to \mathbb{R}$ cts on $[a, b]$, diff on $(a, b) \implies \exists c \in (a, b) : f'(c) = \frac{f(b) - f(a)}{b - a} \iff f(b) = f(a) + f'(c)(b - a)$.
Properties of Derivatives: $f'(x) = 0 \implies f$ const. $f'(x) \geq 0 \implies f$ non-decreasing. $f'(x) a \leq 0 \implies f$ non-increasing.
Dorboux's Theorem: f' has **IVT**: if $a < x_1 < x_2 < b$ and $\exists L \in \mathbb{R} : f'(a) < L < f'(b)$. Then, $\exists x \in (x_1, x_2) : f'(x) = L$.
Partition: $\mathcal{P} \subseteq [a, b] := \{t_j : j = 0, \ldots, n\},\ n \geq 1 : a = t_0 < t_1 < \cdots < t_n = b$.

Dorboux Sums: $\mathcal{U}(f,\mathcal{P}) := \sum_{j=1}^{n} \sup\{f(x) : x \in [t_{j-1}, t_j]\}(t_j - t_{j-1})$. $\mathcal{L}(f,\mathcal{P}) := \sum_{j=1}^{n} \inf\{f(x) : x \in [t_{j-1}, t_j]\}(t_j - t_{j-1})$.
Order: $\forall \mathcal{P} \subseteq [a,b], t_j^* \in [t_{j-1}, t_j], j = 1, \ldots, n,\ \mathcal{L}(f,\mathcal{P}) \leq \mathcal{R}(f,\mathcal{P}) \leq \mathcal{U}(f,\mathcal{P})$.
Monotonicity/Common Refinement: $\mathcal{P}, \mathcal{P}' \subseteq [a,b] : \mathcal{P} \subseteq \mathcal{P}' \implies \mathcal{U}(f,\mathcal{P}') \leq \mathcal{U}(f,\mathcal{P}) \wedge \mathcal{L}(f,\mathcal{P}) \leq \mathcal{L}(f,\mathcal{P}')$.
Order: $\forall \mathcal{P}', \mathcal{P}'' \subseteq [a,b], \mathcal{L}(f,\mathcal{P}') \leq \mathcal{U}(f,\mathcal{P}'') \iff \mathcal{L}(f,\mathcal{P}') \leq \mathcal{L}(f,\mathcal{P}) \leq \mathcal{U}(f,\mathcal{P}) \leq \mathcal{U}(f,\mathcal{P}'')$.
Upper/Lower Dorboux Int: $\overline{\int_a^b} f(x)dx := \inf_{\mathcal{P}\subset[a,b]} \mathcal{U}(f,\mathcal{P})$. $\underline{\int_a^b} f(x)dx := \sup_{\mathcal{P}\subset[a,b]} \mathcal{L}(f,\mathcal{P})$. ***Note:*** $\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx$.
Integrability: $f : [a,b] \to \mathbb{R}$ is int if $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx \in \mathbb{R}$. Then, $\int_a^b f(x)dx := \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$. (int $\implies$ f bdd on $[a,b]$).
Integrability: $f : [a,b] \to \mathbb{R}$ bdd $\implies$ f int on $[a,b] \iff \forall \varepsilon > 0,\ \exists \mathcal{P}_\varepsilon \subset [a,b] : 0 \leq \mathcal{U}(f,\mathcal{P}_\varepsilon) - \mathcal{L}(f,\mathcal{P}_\varepsilon) < \varepsilon$.
Integrability: $f : [a,b] \to \mathbb{R}$ cts on $[a,b] \implies f$ int on $[a,b]$. **Property:** $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.
Monotonicity: $f, g : [a,b] \to \mathbb{R}$ int and $f(x) \leq g(x)\ \forall x \in [a,b] \implies \int_a^b f(x)dx \leq \int_a^b g(x)dx$.
FTC I: $f : [a,b] \to \mathbb{R}$ cts on $[a,b]$. Let $F : [a,b] \to \mathbb{R},\ F(a) = 0, F(x) = \int_a^x f(t)dt$. Then, F diff on (a,b) and $F'(x) = f(x)\ \forall x \in (a,b)$.
FTC II: $f : [a,b] \to \mathbb{R}$ cts on $[a,b]$, diff on (a,b). If f' int on $[a,b]$, then $\int_a^b f'(x)dx = f(b) - f(a)$.

(Rational Zeroes) Prove $\sqrt{n+1} - \sqrt{n-1}$ is irrational $\forall n \in \mathbb{N}$: Assume $\sqrt{n+1} - \sqrt{n-1}$ is rational. Then $x = \sqrt{n+1} - \sqrt{n-1} \implies x + \sqrt{n-1} = \sqrt{n+1} \implies x^2 + 2(\sqrt{n-1})x + (n-1) = n+1 \implies x^2 - 2 = -2(\sqrt{n-1})x \implies x^4 - 4x^2 + 4 = 4x^2(n-1) \implies x^4 - 4nx^2 + 4 = 0$ but $\pm 1, \pm 2, \pm 4$ don't solve the equation so $\sqrt{n+1} - \sqrt{n-1}$ is irrational.

(Sequence Limit) Given $(x_n) = \frac{n^3-11n+2}{2(n^3-6n)}$, Show $\lim_{n\to\infty} x_n = \frac{1}{2}$:
$|x_n - x| = \left|\frac{n^3-11n+2}{2(n^3-6n)} - \frac{1}{2}\right| = \left|\frac{n^3-11n+2-n^3+6n}{2(n^3-6n)}\right| = \left|\frac{-5n+2}{2(n^3-6n)}\right| = \frac{5n-2}{2(n^3-6n)}$. Then $5n + 2 \leq 6n, n \geq 2,\ 2n^3 - 12n \geq \frac{1}{2}n^3, n \geq 4$, so $n = \max\{2, 4\} = 4$. Now, $\frac{5n-2}{2(n^3-6n)} \leq \frac{6n}{\frac{1}{2}n^3} = \frac{12}{n^2} < \varepsilon$ so $N > \frac{\sqrt{12}}{\sqrt{\varepsilon}}$. Let $\varepsilon > 0$. Take $N = \max\left\{2, 4, \frac{\sqrt{12}}{\sqrt{\varepsilon}}\right\}$. Then $\forall n > N$, from before we get $\left|\frac{n^3-11n+2}{2(n^3-6n)} - \frac{1}{2}\right| < \varepsilon \iff \lim_{n\to\infty} x_n = \frac{1}{2}$.

(Cauchy/Series) Given $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, y_n \geq 0\ \forall n \in \mathbb{N}$, show if $\sum_{n=1}^{\infty} x_n$ converges, $\left\{t_n : t_n := \sum_{n=k}^{\infty} x_k\right\}_{n\in\mathbb{N}}$ converges to 0:
Let $s_n = \sum_{k=1}^{n} x_k$. Since s_n converges to $S \in \mathbb{R}$, (s_n) satisfies the Cauchy Criterion. Therefore, $\forall n \in \mathbb{N}, (t_n)$ is well-defined as $t_{n,N} = \sum_{k=n}^{N} x_k$ also satisfies the Cauchy criterion. Then, $t_n := \lim_{N\to\infty} t_{n,N}$ exists and is finite for every fixed $n \in \mathbb{N}$. Fix $N > n > 1 \in$. Then we have $s_{n-1} + t_{n,N} = s_N$. By ALT, $\lim_{N\to\infty}(s_{n-1} + t_{n,N}) = S \implies t_n = S - s_{n-1}$. Since R.H.S converges to 0, so does t_n.

(Comparison Test) Show if $\sum_{n=1}^{\infty} y_n$ converges and $\exists N \in \mathbb{N} : |x_n| \leq y_n\ \forall n > N$ then $\sum_{n=1}^{\infty} x_n$ converges:
Since $|x_n| \leq y_n \forall n > N$, we have that $\sum_{n=N+1}^{\infty} |x_n| \leq \sum_{n=N+1}^{\infty} y_n \implies \sum_{n=N+1}^{\infty} |x_n|$ converges. Since $\sum_{n=1}^{N} x_n$ is finite, it converges. Then, by the comparison test, $\sum_{n=1}^{\infty} x_n$ converges.

((ε, δ) Cty) Show $f(x) = \frac{2x^2+5x-1}{x+1}$ is cts at $x = 1$:
$\left|\frac{2x^2+5x-1}{x+1} - 3\right| = \left|\frac{2x^2+5x-1-3x-3}{x+1}\right| = \frac{|2x^2+2x-4|}{|x+1|} = \frac{2|x+2||x-1|}{|x+1|}$. Now if $|x-1| < \delta$ then $|x+2| \leq |x-1| + 3 = \delta + 3 \leq 4$ and $|x+1| = |x-1+2| \geq 2 - |x-1| \geq 2 - \delta \geq 1$ if $\delta \leq 1$. Then, for $\delta < 1$, we get $\frac{2|x+2|}{|x+1|}|x-1| < 8\delta$ so $\delta = \frac{\varepsilon}{8}$. Let $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{\varepsilon}{8}\}$. Then $|f(x) - f(1)| = \frac{2|x+2|}{|x+1|}|x-1| \leq 8|x-1| < 8\delta = 8\frac{\varepsilon}{8} = \varepsilon$.

(MVT) Prove that $|\cos x - \cos y| \leq |x-y|\ \forall x, y \in \mathbb{R}$:
Apply MVT: Since $\forall x \in \mathbb{R}$, $\sup_{x\in\mathbb{R}} |\sin x| \leq 1$. So, $\forall y \in \mathbb{R}$ we get: $|\cos x - \cos y| \leq \sup_{x\in\mathbb{R}} |(\cos x)'||x-y| \leq \sup_{x\in\mathbb{R}} |\sin x||x-y| \leq |x-y|$.

(MVT) Suppose f is diff on $\mathbb{R}$ and $f(0) = 1, f(1) = f(2) = 1$. Show that $\exists x \in (0,2) : f'(x) = \frac{1}{2}$:
f diff on $\mathbb{R} \implies f$ cts on $\mathbb{R}$. Apply MVT: $\exists x \in (0,2) : f'(x) = \frac{f(2)-f(0)}{2-0} = \frac{1-0}{2-0} = \frac{1}{2}$. Thus, $f'(x) = \frac{1}{2}$ for some $x \in (0,2)$.

(MVT/Dourboux's Theorem) Suppose f is diff on $\mathbb{R}$ and $f(0) = 1, f(1) = f(2) = 1$. Show that $\exists x \in (0,2) : f'(x) = \frac{1}{7}$:
f diff on $\mathbb{R} \implies f$ cts on $\mathbb{R}$. Apply MVT: $\exists x_1 \in (1,2) : f'(x) = \frac{f(2)-f(1)}{2-1} = \frac{1-1}{2-1} = 0$. So $f'(x_1) = 0$. From above, $\exists x_2 \in (0,2) : f'(x_2) = \frac{1}{2}$. Let $c = \frac{1}{7}$. Clearly, $f'(x_1) = 0 < c = \frac{1}{7} < f'(x_2) = \frac{1}{2}$. By Dorboux's Theorem, $\exists x \in (x_1, x_2) \subset (0,2) : f'(x) = c = \frac{1}{7}. Since (1,2) \subsetneq (0,2)$, we have that $\exists x \in (0,2) : f'(x) = \frac{1}{7}$.

(Differentiability) Suppose f, g diff on (a,b), $f'(x) = g'(x)\ \forall x \in (a,b)$. Show that $f(x) = g(x) + c$ for some $c \in \mathbb{R}$:
$h(x) := f(x) - g(x)$ diff on (a,b) and $h'(x) = f'(x) - g'(x) = 0 \implies h$ is constant on (a,b).

(a) (Induction): Prove $1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}\ \forall n \in \mathbb{N}$:
Base case: n = $1 \to 1 \geq 1$. IH: $1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} \geq \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n}\sqrt{n+1}}{\sqrt{n+1}} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n^2+n}+1}{\sqrt{n+1}} \geq \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$

Using (a), show $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges:
Define $s_n := \sum_{k=1}^{n} \frac{1}{\sqrt{n}}$. From (a), $s_n \geq \sqrt{n}\ \forall n \in \mathbb{N}$. Then $0 \leq \frac{1}{s_n} \leq \frac{1}{\sqrt{n}}$ and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} \to 0 \implies \frac{1}{s_n} \to 0$ by squeeze theorem. Then, $\lim_{n\to\infty} s_n \to +\infty$ so the series diverges.

(Im/Possible) $f : [0,1] \to \mathbb{R}$ s.t. $|f|$ is int on $[0,1]$ but f is not:
$f(x) = 1_{\mathbb{Q}}(x) - 1_{\mathbb{R}\setminus\mathbb{Q}}(x)$. Then $|f| = 1\ \forall x \in \mathbb{R} \implies$ int but f is not int.

(Subsequences) Let (x_n) have the property: $\exists x \in \mathbb{R} : \forall (x_{n_k}), \exists (x_{n_{k_l}}) \to x$. Show $(x_n) \to x$:
Assume by contradiction $(x_n) \not\to x$. Then, $\exists (x_{n_k}) : (x_{n_k}) \not\to x$. i.e., $\exists \varepsilon_0 > 0 : \forall N \in \mathbb{N}$, $\exists n > N : |x_n - x| \geq \varepsilon_0$. Take $N = 1$ and get an $n_1 > 1$ for which $|x_{n_1} - x| \geq \varepsilon_0$. Then, set $N = \max\{2, n_1\}$ and get $n_2 > N : |x_{n_2} - x| \geq \varepsilon_0$. Continue inductively to get $|x_{n_k} - x| \geq x_0 \ \forall k \in \mathbb{N}$. Hence, any subsequence of this subsequence will satisfy the above and won't converge to x, a contradiction.

Assume $f : \mathbb{R} \to \mathbb{R}$ has the property: $\lim_{n\to\infty} f(2^{-n}) = f(0)$. Is f cts at 0:
No: $f(x) = \begin{cases} 1 & , x \in \{0\} \cup \{2^{-n} : n \in \mathbb{N}\} \\ 2 & otherwise \end{cases}$. Then if $x_n = 2^{-n}$, $\lim_{n\to\infty} f(x_n = 2^{-n}) = f(0) = 1$ but $(y_n) \subset \mathbb{I} : y_n \to 0 asn \to \infty$, then $\lim_{n\to\infty} f(y_n) = 2$.
Then $\lim_{x\to 0} f(x)$ DNE $\implies$ not cts at 0.

(Derivative) Calculate the derivative of $f(x) = \frac{3x+4}{2x-1}$ at $x = 1$:
$\lim_{x\to 1} \frac{\frac{3x+4}{2x-1} - \frac{(3)(1)+4}{(2)(1)-1}}{x-1} = \lim_{x\to 1} \frac{\frac{3x+4-14x+7}{2x-1}}{x-1} = \lim_{x\to 1} \frac{\frac{11-11x}{2x-1}}{x-1} = \lim_{x\to 1} \frac{-11(x-1)}{(2x-1)(x-1)} = \lim_{x\to 1} \frac{-11}{2x-1} = \lim_{x\to 1} \frac{-11}{2(1)-1} = -11.$

Let $f, g : [0,1] \to \mathbb{R}$ be defined by: $f(x) = \begin{cases} 1 & x \geq \frac{1}{2} \\ -1 & x < \frac{1}{2} \end{cases}$, $g(x) = \begin{cases} 1 & x \geq \frac{1}{2} \\ -1 & x < \frac{1}{2} \end{cases}$. Show f is upper semi-cts on $[0,1]$ but g is not:
W.T.S: Given $x \in [0,1], \varepsilon > 0$, $\exists \delta > 0 : |y - x| < \delta \implies f(y) < f(x) + \varepsilon$.
Let $x \in [0,1]$. $x < \frac{1}{2} \implies f(x) = -1$. Let $\varepsilon > 0$. Take $\delta = \min\left\{\frac{1}{2} - x, \varepsilon\right\}$. Then, whenever $|y - x| < \delta$, there are two cases. $y < \frac{1}{2} \implies f(y) = -1$ so $f(y < f(x) + \varepsilon$. $y \geq \frac{1}{2} \implies f(y) = 1$ so $f(y) < f(x) + \varepsilon$. In both cases, f is upper semi-cts on $[0,1]$.
Take $x = \frac{1}{2}$ and $\varepsilon = 1$. Then $\forall \delta > 0$, $\exists y \in [0,1] : |y - x| < \delta$ but $g(y) = 1, g(x) = -1$. So, $\nexists \delta > 0 : g(y) < g(x) + \varepsilon \implies g(x)$ not upper semi-cts on $[0,1]$.

(sup/inf) Show $\sup A - \inf B = \sup\{a - b : a \in A, b \in B\}$:
By LUBP, $\sup A, \inf B$ exist. Then, we have $\forall a, b \in A, B, a \leq \sup A, b \geq \inf B$. So, $\forall a, b \in A, B,\ a - b \leq \sup A - \inf B \implies \sup(A - B) \leq \sup A - \inf B$, so $\sup A - \inf B$ is an upper bound for $A - B$. Let $\varepsilon > 0$. Then, $\exists a \in A : \sup A - \frac{\varepsilon}{2} < a$, $\exists b \in B : \inf B + \frac{\varepsilon}{2} > b$. Let α be an upper bound for $A - B$. Then, $a - b \leq \alpha \ \forall a \in A,\ b \in B \implies \sup A - \inf B < \alpha + \varepsilon \implies \sup A - \inf B \leq \alpha$, so $\sup A - \inf B \leq \sup(A - B)$. Thus, $\sup A - \inf B = \sup\{a - b : a \in A, b \in B\}$.

(Dorboux Sums) Show $\forall \mathcal{P} \subset [a,b]$, we have $\mathcal{U}(f^2, \mathcal{P}) - \mathcal{L}(f^2, \mathcal{P}) \leq 2M[\mathcal{U}(f, \mathcal{P}) - \mathcal{L}(f, \mathcal{P})]$:

$$\begin{aligned}
\mathcal{U}(f^2, \mathcal{P}) - \mathcal{L}(f^2, \mathcal{P}) &= \sum_{j=1}^{n} \sup\left\{f^2(x) : x \in [t_{j-1}, t_j]\right\}(t_j - t_{j-1}) - \sum_{j=1}^{n} \inf\left\{f^2(y) : y \in [t_{j-1}, t_j]\right\}(t_j - t_{j-1}) \\
&= \sum_{j=1}^{n} \sup\left\{f^2(x) - f^2(y) : x, y \in [t_{j-1}, t_j]\right\}(t_j - t_{j-1}) && \text{from (a)} \\
&= \sum_{j=1}^{n} \sup\left\{[f(x) + f(y)][f(x) - f(y)] : x, y \in [t_{j-1}, t_j]\right\}(t_j - t_{j-1}) \\
&\leq \sum_{j=1}^{n} \sup\left\{[|f(x)| + |f(y)|][f(x) - f(y)] : x, y \in [t_{j-1}, t_j]\right\}(t_j - t_{j-1}) \\
&\leq \sum_{j=1}^{n} \sup\left\{[M + M][f(x) - f(y)] : x, y \in [t_{j-1}, t_j]\right\}(t_j - t_{j-1}) && f \text{ is bounded} \implies |f(x)| \leq M \forall x \in [a,b] \\
&= \sum_{j=1}^{n} \sup\left\{2M[f(x) - f(y)] : x, y \in [t_{j-1}, t_j]\right\}(t_j - t_{j-1}) \\
&= 2M \sum_{j=1}^{n} \sup\left\{[f(x) - f(y)] : x, y \in [t_{j-1}, t_j]\right\}(t_j - t_{j-1}) \\
&= 2M[\mathcal{U}(f, \mathcal{P}) - \mathcal{L}(f, \mathcal{P})]
\end{aligned}$$

(Integrability) Prove if $f : [a,b] \to \mathbb{R}$ is integrable, so is f^2:
f is int $\implies \forall \varepsilon > 0$, $\exists \mathcal{P}_\varepsilon \subset [a,b] : 0 \leq \mathcal{U}(f, \mathcal{P}_\varepsilon) - \mathcal{L}(f, \mathcal{P}_\varepsilon) < \frac{\varepsilon}{2M}$. Then, $\mathcal{U}(f^2, \mathcal{P}_\varepsilon) - \mathcal{L}(f^2, \mathcal{P}_\varepsilon) \leq 2M[\mathcal{U}(f, \mathcal{P}) - \mathcal{L}(f, \mathcal{P})] \leq 2M\frac{\varepsilon}{2M} = \varepsilon$. So f^2 int.

(IVT) Let $a < b$ and $f : [a,b] \to [a,b]$ be cts on $[a,b]$. Show that $\exists c \in [a,b] : f(c) = c$:
Define $g(x) := f(x) - x$. Then g is cts on $[a,b]$ and $g(a) = f(a) - a \geq 0$, $g(b) = f(b) - b \leq 0$. If either $g(a), g(b) = 0$, we are done. Else, $g(a) > 0 \wedge g(b) < 0$. Then by IVT, $\exists c \in (a,b) : g(c) = 0 \implies f(c) = c$.

(IVT) Let f and g be cts functions on $[a,b] : f(a) \geq g(a) \wedge f(b) \leq g(b)$. Prove $\exists x_0 \in [a,b] : f(x_0) = g(x_0)$:
Define $h(x) := f(x) - g(x)$. If either $f(a) \vee f(b) = g(a) \vee g(b)$, we are done. Else, $h(a) = f(a) - g(a) < 0 \wedge h(b) = f(b) - g(b) > 0$. f, g cts on $[a,b] \implies h(x)$ cts on $[a,b]$. Then by IVT, $\exists x_0 \in (a,b) : h(x_0) = 0 \implies f(x_0) = g(x_0)$.

(IVT) Let $f : \mathbb{R} \to \mathbb{R}$ cts on $\mathbb{R}$, inc on $[-1, 0$. Assume $f(0) = 1$. Show $\exists x \in [-1, 0) : f(x) = -x^3$:
Define $g(x) := f(x) - (-x^3) = f(x) + x^3$. Then g is cts on $[-1, 0]$. We have $g(0) = f(0) + 0 = 1$. Since f is increasing on $[-1, 0)$, $f(-1) \leq f(0) = 1 \implies g(-1) = f(-1) - 1 \leq 1 - 1 = 0$. So if $g(-1) = 0$, put $x_0 = -1$. Otherwise, $g(-1) < 0 < g(0)$. Apply IVT to get $x_0 \in (-1, 0) : g(x_0) = 0$.

(Partitions) Let $f, g : [a,b] \to \mathbb{R}$ be int on $[a,b]$ and s.t. $f(x) = g(x) \ \forall x \in [a,b] \cap \mathbb{Q}$. Show that for any $\mathcal{P} \subset [a,b]$, we have $\mathcal{L}(f, \mathcal{P}) \leq \mathcal{U}(f, \mathcal{P})$:
Let $I \subseteq [a,b]$. Then, $\inf_{x\in I} f(x) \leq \inf_{x \in I\cap\mathbb{Q}} f(x) = \inf_{x\in I\cap\mathbb{Q}} g(x) \leq \sup_{x\in I\cap\mathbb{Q}} g(x) \leq \sup_{x\in I} g(x)$.

(Uni Cty) Determine if $f(x) = \log x$ is uni cts on $(0, 1)$:
No: $x_n = e^{-n}, y_n = e^{-2n}$. Then $|x_n - y_n| \to 0$ as $n \to \infty$ but $|f(x_n) - f(y_n)| = |\log e^{-n} - \log e^{-2n}| = |2n - n| = |n| \geq 1$.

(Uni Cty) Determine if $f(x) = \sin \frac{1}{x^2}$ is uni cts on $(0, 1]$:
No: $x_n = \frac{1}{\sqrt{\frac{\pi}{2}+\pi n}}$ Then $x_n \to 0$ as $n \to \infty \implies$ Cauchy but $f(x_n) = \sin \frac{1}{x_n^2} = \sin \frac{1}{\left(\frac{1}{\sqrt{\frac{\pi}{2}+\pi n}}\right)} = \sin\left(\frac{\pi}{2} + \pi n\right) = (-1)^n$ which is not Cauchy.

(Uni Cty) Determine if $f(x) = \frac{1}{x-3}$ is uni cts on $(4, \infty)$:
Yes: Let $\varepsilon > 0$. Let $\delta = \min\{1, \varepsilon\}$. Then, $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. $|f(x) - f(y)| = \left|\frac{1}{x-3} - \frac{1}{y-3}\right| = \frac{|y-x|}{|x-3||y-3|} \leq \frac{|x-y|}{(1)(1)} = |x - y| < \delta = \varepsilon$.

(Limits) Find the limit of $\lim\limits_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}, a > 0$:
$\lim\limits_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} = \lim\limits_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{\sqrt{x}^2-\sqrt{a}^2} = \lim\limits_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a})} = \lim\limits_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}} = \frac{1}{\sqrt{a}+\sqrt{a}} = \frac{1}{2\sqrt{a}}$.

(Series) Study the convergence of $\sum\limits_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$:

Comparison Test: $x_n = \frac{\cos^2 n}{n^2}$, $y_n = \frac{1}{n^2}$. Then, $0 \leq x_n \leq y_n\ \forall n \in \mathbb{N}$. $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges $\implies \sum\limits_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$ converges.

(Series) Study the convergence of $\sum\limits_{n=2}^{\infty} \frac{1}{\log n}$:

Comparison Test: For $n \geq 10$, $\log n \leq n \implies \frac{1}{\log n} \geq \frac{1}{n}$. So $x_n = \frac{1}{\log n} \geq y_n = \frac{1}{n}\ \forall n \geq 10 \in \mathbb{N}$. Then, $\sum\limits_{n=2}^{\infty} \frac{1}{n}$ diverges $\implies \sum\limits_{n=2}^{\infty} \frac{1}{\log n}$ diverges.

(Series) Study the convergence of $\sum\limits_{n=1}^{\infty} \frac{n!}{n^n}$:
Ratio Test: $\lim\limits_{n\to\infty} \left|\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{(n+1)^{n+1}}{(n+1)!}\right| = \lim\limits_{n\to\infty} \left|\frac{n^n}{(n+1)^n}\right| = \lim\limits_{n\to\infty} \left|\frac{1}{(1+\frac{1}{n})^n}\right| = \frac{1}{e} < 1 \implies$ converges.

(Integrability) Find $f : |f|$ is int but f is not:
$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$.Let $\mathcal{P} := \left\{t_j : t_j = \frac{j}{n}\right\}$. Then, we have

$$\begin{aligned}
\mathcal{U}(f, \mathcal{P}) &= \sum_{j=1}^{n} \sup\{f(x) : x \in [t_{j-1}, t_j]\}(t_j - t_{j-1}) \\
&= \sum_{j=1}^{n} M(f, [t_{j-1}, t_j])(t_j - t_{j-1}) \\
\mathcal{U}(f, \mathcal{P}) &= \sum_{j=1}^{n} 1 \cdot (t_j - t_{j-1}) = 1
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}(f, \mathcal{P}) &= \sum_{j=1}^{n} \inf\{f(x) : x \in [t_{j-1}, t_j]\}(t_j - t_{j-1}) \\
&= \sum_{j=1}^{n} m(f, [t_{j-1}, t_j])(t_j - t_{j-1}) \\
\mathcal{L}(f, \mathcal{P}) &= \sum_{j=1}^{n} -1 \cdot (t_j - t_{j-1}) = 1
\end{aligned}$$

thus $\mathcal{U}(f, \mathcal{P}) = 1 \neq -1 = \mathcal{L}(f, \mathcal{P})$ so f is not int.
For $|f|$, we have

$$\begin{aligned}
\mathcal{U}(|f|, \mathcal{P}) &= \sum_{j=1}^{n} \sup\{|f(x)| : x \in [t_{j-1}, t_j]\}(t_j - t_{j-1}) \\
&= \sum_{j=1}^{n} M(|f|, [t_{j-1}, t_j])(t_j - t_{j-1}) \\
\mathcal{U}(|f|, \mathcal{P}) &= \sum_{j=1}^{n} 1 \cdot (t_j - t_{j-1}) = 1
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}(|f|, \mathcal{P}) &= \sum_{j=1}^{n} \inf\{|f(x)| : x \in [t_{j-1}, t_j]\}(t_j - t_{j-1}) \\
&= \sum_{j=1}^{n} m(|f|, [t_{j-1}, t_j])(t_j - t_{j-1}) \\
\mathcal{L}(|f|, \mathcal{P}) &= \sum_{j=1}^{n} 1 \cdot (t_j - t_{j-1}) = 1
\end{aligned}$$

thus $\mathcal{U}(|F|, \mathcal{P}) = 1 = 1 = \mathcal{L}(|f|, \mathcal{P})$ so $|f|$ is int.