Partial Order: $\forall x, y, z \in A$: Reflexive: $x\mathcal{R}x$, Anti-symmetric: $x\mathcal{R}y, y\mathcal{R}x \implies x = y$, Transitive: $x\mathcal{R}y, y\mathcal{R}z \implies x\mathcal{R}z$.
Total Order: $\forall x, y \in A, x\mathcal{R}y \vee y\mathcal{R}x$
Equivalence Relation: $\forall x, y, z \in A$: Reflexive: $x\mathcal{R}x$, Symmetric: $x\mathcal{R}y = y\mathcal{R}x$, Transitive: $x\mathcal{R}y, y\mathcal{R}z \implies x\mathcal{R}z$.
Equivalence Class: $[x] := \{y \in A : x \sim y\}$
Induction: ***(i)*** *P_1 is true.* ***(ii)*** *Assume P_n is true for some $n \in \mathbb{N}$. Prove P_{n+1} is true.* Then, P_n is true $\forall n \in \mathbb{N}$.
Ordered Fields: A field with a partial order $(\leq)$ s.t.: ***(i)*** If $x, y, z \in \mathbb{F},\ x < y \implies x + z < x + y$, ***(ii)*** $x, y \in \mathbb{F},\ x, y > 0 \implies xy > 0$
Algebraic Number: a is algebraic if it solves $c_n x^n + \cdots + c_1 x + c_0 = 0$ for some $n \in \mathbb{N},\ c_0, c_n \in \mathbb{Z}, c_n \neq 0$ (e.g. $\sqrt[n]{2}$. ***Note:*** $\mathbb{Q} \subset \{algebraic\ numbers\}$)
Rational Zeros Theorem: *Suppose $c_0, \cdots, c_n \in \mathbb{Z},\ r \in \mathbb{Q}$ satisfies $c_n r^n + \cdots + c_1 r + c_0 = 0$ for some $n \in \mathbb{N},\ c_n \neq 0$. Let $r = \frac{c}{d}, c, d \in \mathbb{Z}, d \neq 0$, be coprime. Then $c,\ d$ divides c_0, c_n.*
LUBP: Given $A \subseteq \mathbb{E}$ where $\mathbb{E}$ is an ordered set, $\exists \sup A \in \mathbb{E} \iff A \neq \emptyset,\ A \subseteq \mathbb{E}$, A is bounded above. $\sup A := \alpha,\ \exists \alpha, \beta \in \mathbb{E}$ s.t. $\forall a \in A,\ a \leq \alpha \leq \beta$.
GLBP: Given $A \subseteq \mathbb{E}$ where $\mathbb{E}$ is an ordered set, $\exists \inf A \in \mathbb{E} \iff A \neq \emptyset,\ A \subseteq \mathbb{E}$, A is bounded below. $\inf A := \alpha,\ \exists \alpha, \beta \in \mathbb{E}$ s.t. $\forall a \in A,\ \beta \leq \alpha \leq a$.
Archemedian Property: If $y \in \mathbb{R},\ x > 0$, then $\exists n \in \mathbb{N}$ s.t. $n \cdot x > y$. Put $x = 1 : \exists n \in \mathbb{N}$ s.t. $n > y$. Put $y = 1 : \exists n \in \mathbb{N}$ s.t. $n \cdot x > 1 \rightsquigarrow 0 < \frac{1}{n} < x$.
Density of $\mathbb{Q}$ in $\mathbb{R}$: $\forall x, y \in \mathbb{R} : x < y, \exists p \in \mathbb{Q} : x < p < y$
Sequence: A function $f : \mathbb{N} \to \mathbb{R} \iff n \mapsto f(n) \iff n \mapsto f_n$ e.g. $(1, \frac{1}{2}, \frac{1}{3}, \cdots)$, $x_n = \frac{1}{n}\ \forall n \in \mathbb{N}$, $\{x_n : n \in \mathbb{N}\}$, $(x_n)_{n=1}^{\infty}$, $(x_n)_{n \in \mathbb{N}}$
Convergent: A sequence (x_n) converges to $x \in \mathbb{R}$ if: $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N,\ |x_n - x| < \varepsilon$. We write $(x_n) \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n := x$, where x is the **limit** of (x_n).
Divergent: A sequence that does not **converge**.
Absolute Value: $|x| = \{x \text{ if } x \geq 0\},\ \{-x \text{ if } x < 0\} \implies |x| \geq 0$. ***(i)*** $|xy| = |x||y|$, ***(ii)*** $|x - y| \leq z \iff z \leq x - y \leq z \iff y - z \leq x \leq y + z$
Triangle Inequality: $|x + y| \leq |x| + |y| \implies |x - y| = |x + (-z + z) - y| \leq |x - z| + |z - y|\ \forall x, y, z \in \mathbb{R}$.
Unique Limits: $x_n \to x,\ x_n \to y \implies x = y$. $|x - y| = |x + (-x + x) - y| \leq |x_n - x| + |x_n - y| = \varepsilon$ if $|x_n - x|, |x_n - y| \leq \frac{\varepsilon}{2}$.
Algebraic Limit Theorem: $x_n \to x, y_n \to y \implies$ ***(i)*** $ax_n \to ax$, ***(ii)*** $x_n \pm y_n \to x \pm y$, ***(iii)*** $x_n \cdot y_n \to x \cdot y$ ***(iv)*** $\frac{x_n}{y_n} \to \frac{x}{y}, y \neq 0$

$\subseteq$ defines a Partial (not Total) Order on $\mathcal{P}(A)$:
Reflexive: $B \in \mathcal{P}(A)$. $B = B \implies B \subseteq B$.
Anti-symmetric: $B, C \in \mathcal{P}(A)$. $(B \subset C) \wedge (C \subset B) \implies B = C$.
Transitivity: $B, C, D \in \mathcal{P}(A) : B \subset C \subset D$. $x \in B \implies x \in C \implies x \in D \implies B \subset D$.
$\mathcal{P}(A)$ is not a Total Order: $A = \{a, b\} \implies \mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. $\{a\} \subset \{b\}$ and $\{b\} \subset \{a\}$

$(2 + \sqrt{3})^{\frac{1}{3}}$ is irrational: Assume $(2 + \sqrt{3})^{\frac{1}{3}}$ is rational. Then
$x = (2 + \sqrt{3})^{\frac{1}{3}} \implies x^3 = 2 + \sqrt{3} \implies (x^3 - 2)^2 = (\sqrt{3})^2 \implies x^6 - 4x^3 + 4 = 3 \implies x^6 - 4x^3 + 1 = 0$ but ± 1 does not solve the equation so $(2 + \sqrt{3})^{\frac{1}{3}}$ is irrational.

Show $\sup\{A := \{p \in \mathbb{Q} : p < r\}\} = r$ **where** $r \in \mathbb{R}$**:** r is an upper bound for A and $A \neq \emptyset$ by the **Archemedian Property** (applied to $-r$). So $\sup A$ exists in $\mathbb{R}$. By definition, $\sup A \leq r$. Assume by contradiction $\sup A < r$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists q \in \mathbb{Q} : \sup A < q < r \implies q \in A$ which is a contradiction.

Prove $(1 + x)^n \geq 1 + nx \forall n \in \mathbb{N}$**:** ***(i)*** $P_1 : 1 + x \geq 1 + x$
(ii) Assume P_n is true for some $n \in \mathbb{N}$

$$\begin{aligned}(1 + x)^{n+1} = (1 + x)^n(1 + x) &\geq (1 + nx)(1 + x) \\ &\geq 1 + nx + x + nx^2 \\ &\geq 1 + (n + 1)x \geq 1 + (n + 1)x + nx^2 \\ (1 + x)^{n+1} &\geq 1 + (n + 1)x\end{aligned}$$

Define $x\mathcal{R}y : x - y = 2k,\ k \in \mathbb{Z}$**. Prove it's an equivalence relation. How many unique classes?**
Reflexive: $x \in \mathbb{Z},\ x - x = 0 = 2 \cdot 0,\ 0 \in \mathbb{Z} \implies x\mathcal{R}x$.
Symmetric: $x, y \in \mathbb{Z} : x\mathcal{R}y \implies \exists k \in \mathbb{Z} : x - y = 2 \cdot k$. Then $y - x = -(x - y) = -(2 \cdot k) = -k,\ -k \in \mathbb{Z} \implies y\mathcal{R}x$.
Transitive: $x, y, z \in \mathbb{Z} : x\mathcal{R}y \wedge y\mathcal{R}z \implies \exists k_1, k_2 \in \mathbb{Z} : x - y = 2k_1 \wedge y - z = 2k_2$. Then
$x - z = x + (-y + y) - z = (x - y) + (x - z) = 2k_1 + 2k_2 = 2(k_1 + k_2),\ k_1, k_2 \in \mathbb{Z} \implies x\mathcal{R}z$.
There are exactly 2 unique **Equivalence Classes**:
$[1] := \{x \in \mathbb{Z} : x\mathcal{R}1\}$. $x \in [1] \implies \exists k \in \mathbb{Z} : x - 1 = 2k \iff x = 2k + 1 \implies x$ is **odd**.
$[2] := \{x \in \mathbb{Z} : x\mathcal{R}2\}$. $x \in [2] \implies \exists k \in \mathbb{Z} : x - 2 = 2k \iff x = 2(k + 1) \implies x$ is **even**.

Prove $\sqrt{n + 1} - \sqrt{n - 1}$ **is irrational** $\forall n \in \mathbb{N}$**:** Assume $\sqrt{n + 1} - \sqrt{n - 1}$ is rational. Then
$x = \sqrt{n + 1} - \sqrt{n - 1} \implies x + \sqrt{n - 1} = \sqrt{n + 1} \implies x^2 + 2(\sqrt{n - 1})x + (n - 1) = n + 1 \implies x^2 - 2 = -2(\sqrt{n - 1})x \implies x^4 - 4x^2 + 4 = 4x^2(n - 1) \implies x^4 - 4nx^2 + 4 = 0$ but $\pm 1, \pm 2, \pm 4$ don't solve the equation so $\sqrt{n + 1} - \sqrt{n - 1}$ is irrational.

$Y := \{mx + b : x \in X, m, b \in \mathbb{R}^+\}$**. Prove** $\sup Y = m \sup X + b$**:** From (a), $\sup Y \leq m \sup X + b$. Let $z \in \mathbb{R} : z < m \sup X + b$. Since $m > 0$, $z - b < m \sup X \iff \frac{z - b}{m} < \sup X$. By definition, $\frac{z - b}{m}$ is not an upper bound for $X \implies \exists x \in X : \frac{z - b}{m} < x \implies z < mx + b \in Y$. Therefore, z is not an upper bound for Y.

Show $x_n = \frac{\sqrt{n}+1}{\sqrt{n+1}} \forall n \in \mathbb{N}$ **converges (to 1):**

$$\begin{aligned}\left|\frac{\sqrt{n}+1}{\sqrt{n+1}} - 1\right| &= \left|\frac{\sqrt{n}+1-\sqrt{n+1}}{\sqrt{n+1}}\right| \\ &\leq \frac{|\sqrt{n}-\sqrt{n+1}|}{\sqrt{n+1}} + \frac{1}{\sqrt{n+1}} \\ &= \frac{1}{(\sqrt{n+1}+\sqrt{n})\sqrt{n+1}} + \frac{1}{\sqrt{n+1}} \\ &\leq \frac{1}{n+1} + \frac{1}{\sqrt{n+1}} \\ &\leq \frac{2}{\sqrt{n}} < \varepsilon \\ n &> \frac{4}{\varepsilon^2}\end{aligned}$$

Prove $\lim \frac{2n-1}{3n+2} = \frac{2}{3}$

$$\begin{aligned}\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| &< \varepsilon \\ \left|\frac{6n-3-(6n+4)}{3(3n+2)}\right| &< \varepsilon \\ \left|\frac{-7}{3(3n+2)}\right| &< \varepsilon \\ \frac{7}{3(3n+2)} &< \varepsilon \\ \frac{7}{9n+6} &< \varepsilon \\ n &> \frac{7-6\varepsilon}{9\epsilon}\end{aligned}$$

note: $\left|\frac{-7}{3(3n+2)}\right| \leq \frac{7}{3(3n+2)}$

Let $\varepsilon > 0$. Let $N \geq \frac{7-6\varepsilon}{9\epsilon}$. Then $\forall n > N$, we have

$$n > \frac{7-6\varepsilon}{9\epsilon} \implies \left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| < \varepsilon$$

Prove $\lim \frac{n+6}{n^2-6} = 0$

$$\begin{aligned}\left|\frac{n+6}{n^2-6} - 0\right| &< \varepsilon \\ \left|\frac{n+6}{n^2-6}\right| &< \varepsilon\end{aligned}$$

Note that when $n \geq 6$, we have that $|n+6| \leq 2n$, $|n^2-6| \geq \frac{1}{2}n^2$.

$$\begin{aligned}\left|\frac{n+6}{n^2-6}\right| \leq \frac{2n}{\frac{1}{2}n^2} &< \varepsilon \\ \frac{4n}{n^2} &< \varepsilon \\ \frac{4}{n} &< \varepsilon \\ n &> \max\{\frac{4}{\varepsilon}, 6\}\end{aligned}$$

Let $\varepsilon > 0$. Let $N \geq \max\{\frac{4}{\varepsilon}, 6\}$. Then $\forall n > N$, we have

$$n > \max\{\frac{4}{\varepsilon}, 6\} \implies \left|\frac{n+6}{n^2-6}\right| \leq \frac{2n}{\frac{1}{2}n^2} < \varepsilon$$

Prove $\sup(A_1 \cup A_2) = \max\{\sup A_1, \sup A_2\} \implies \sup\left(\bigcup_{k=1}^n A_k\right) = \max_{k=1,\dots,n}\{\sup A_k\}$**:** By **LUBP** of $\mathbb{R}$, $\sup A_{1,2}$ exist $\implies A_1 \cup A_2 \implies \sup(A_1 \cup A_2) \implies \sup(A_1 \cup A_2) \leq \max\{\sup A_1, \sup A_2\}$. $\impliedby \sup A_i \leq \sup(A_1 \cup A_2)$, $i = 1, 2$. Then, $\sup\left(\bigcup_{k=1}^n A_k\right) \leq \max_{k=1,\dots,n}\{\sup A_k\}$. $\impliedby \max_{k=1,\dots,n}\{\sup A_k\} \leq \sup\left(\bigcup_{k=1}^n A_k\right), k = 1, \dots, n$.