Monotone Convergence Theorem: Monotone inc/dec and bounded above/below $\implies (x_n)$ converges.
Bolzono-Weirstrauss Theorem: Bounded $\implies \exists(x_{n_k})$ that converges.
Squeeze Theorem: Given $(x_n),\ (y_n),\ (z_n) : y_n \leq x_n \leq z_n \forall n \in \mathbb{N}$ and $y_n \to x,\ z_n \to x$ as $n \to \infty$, $x_n \to x$ as $n \to \infty$.
Test for Divergence: $(x_n) \not\to 0 \implies \sum x_n$ does not converge.
Cauchy Sequence: $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n, m > N, |x_n - x_m| < \varepsilon$. ***Note:*** (x_n) is cauchy $\iff (x_n)$ converges in $\mathbb{R}$ only.
Geometric Series: Given $x \in \mathbb{R}$, $S_n = \sum\limits_{k=1}^{n} x^k = \frac{1-x^{n+1}}{1-x}$ if $x \neq 1$. $|x| < 1 \implies S_n \to \frac{1}{1-x} \implies (x)^n \to 0$ by ALT. $|x| > 1 \implies S_n \to +\infty$.
Comparison Test: Assume $y_n \geq 0\ \forall n \geq N$. If $|x_n| \leq y_n \forall n \in \mathbb{N}$, then:
(i) $\sum y_n$ converges $\implies \sum x_n$ converges.
(ii) $\sum |x_n|$ diverges $\implies \sum y_n$ diverges.
(iii) $\sum y_n \to +\infty$ & $x_n \geq y_n, \forall n \in \mathbb{N} \implies \sum x_n \to +\infty$.
Absolute Convergence Test: $\sum |x_n|$ converges $\implies \sum x_n$ converges.
Cauchy Condensation Test: Given (x_n) decreasing and nonnegative, $\sum\limits_{n=1}^{\infty} x_n$ converges $\iff \sum\limits_{n=1}^{\infty} 2^n x_{2^n}$ converges.
Cauchy Criterion: $\sum\limits_{n=1}^{\infty} x_n$ converges $\iff \forall \varepsilon > 0, \exists N \in \mathbb{N} : n > m \geq N \implies |x_{m+1} + \cdots + x_n| < \varepsilon$.
p-series Test: $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff p > 1$.
Ratio Test: Given $x_n \neq 0$, $\lim\limits_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = L$ converges absolutely if $L < 1$, diverges if $L > 1$, inconclusive if $L = 1$.
Root Test: Given x_n, $\lim\limits_{n\to\infty} |x_n|^{\frac{1}{n}} = L$ converges absolutely if $L < 1$, diverges if $L > 1$, inconclusive if $L = 1$.
Alternating Series Test: If a sequence (x_n) is decreasing and converges to 0, then $\sum(-1)^{n+1}x_n$ converges.
Exponent rules with e**:** $x^a = e^{a \log x}$
Existance of Limits: $\lim\limits_{x\to c} f(x)$ exists $\iff \lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x)$
Continuity (ε, δ)**:** $f : A \to \mathbb{R}$ is continuous at $c \in A$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. whenever $x \in A, |x - c| < \delta$, we have $f(x) - f(c)| < \varepsilon$.
Functional Limit (ε, δ)**:** $\lim\limits_{x\to c} f(x) = L \iff$ for $c \in L_A$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. whenever $x \in A, 0 < |x - c| < \delta$, we have $f(x) - L| < \varepsilon$.

Prove using the (ε, δ) **definition that** $f(x) = \sqrt{x + \sqrt{x}}$ **is continuous on** $[0, +\infty)$**.**
Scratch: Case 1: $c = 0$. Then, $|f(x) - f(0)| = \sqrt{x + \sqrt{x}} \leq \sqrt{2\sqrt{x}} = 2^{\frac{1}{x}} x^{\frac{1}{4}}$. $x < \delta \implies \delta \leq \frac{1}{4}\varepsilon^4$.
Proof: Let $\varepsilon > 0$. Choose $\delta = \min\left\{1, \frac{1}{4}\varepsilon^4\right\}$. Then, $|f(x) - f(0)| = \sqrt{x + \sqrt{x}} \leq \sqrt{2\sqrt{x}} = 2^{\frac{1}{x}} x^{\frac{1}{4}} < \varepsilon$ whenever $0 \leq x < \delta$.
Scratch Case 2: $c \neq 0$. Then, we have:

$$\begin{aligned}
0 < |f(x) - f(c)| &= \left|\sqrt{x + \sqrt{x}} - \sqrt{c + \sqrt{c}}\right| \\
&= \frac{|x + \sqrt{x} - c - \sqrt{c}|}{|\sqrt{x + \sqrt{x}} + \sqrt{c + \sqrt{c}}|} \leq \frac{|x - c + \sqrt{x} - \sqrt{c}|}{|\sqrt{c + \sqrt{c}}|} \\
&= \frac{|x - c + \frac{x-c}{\sqrt{x}+\sqrt{c}}|}{|\sqrt{c + \sqrt{c}}|} \leq \frac{|x - c| + \left|\frac{x-c}{\sqrt{c}}\right|}{|\sqrt{c + \sqrt{c}}|} \\
&= \frac{(|x - c|)\left(\sqrt{c} + 1\right)}{\sqrt{c}\sqrt{c + \sqrt{c}}} < \varepsilon \\
&\implies \delta = \frac{\sqrt{c}\sqrt{c + \sqrt{c}}}{\sqrt{c} + 1}\varepsilon
\end{aligned}$$

Proof: Let $\varepsilon > 0$. Choose $\delta = \frac{\sqrt{c}\sqrt{c+\sqrt{c}}}{\sqrt{c}+1}\varepsilon$. By above, we have $|f(x) - f(c)| < \frac{\sqrt{c}+1}{\sqrt{c}\sqrt{c+\sqrt{c}}}\delta = \varepsilon$

Let $(x_n),\ (y_n),\ (z_n)$ **be sequences of real numbers such that there exists** $N_0 \in \mathbb{N}$ **for which** $y_n \leq x_n \leq z_n$ **for all** $n > N_0$**. If the series** $\sum\limits_{n=1}^{\infty} y_n$ **and** $\sum\limits_{n=1}^{\infty} z_n$ **converge, show that the series** $\sum\limits_{n=1}^{\infty} x_n$ **converges.**
Proof: Let $\varepsilon > 0$. Since the series $\sum y_k,\ \sum z_k$ converge, they satisfy the Cauchy criterion, so $\exists N_1, N_2 \in \mathbb{N}$ s.t.

$$\left|\sum_{k=m+1}^{n} y_k\right| < \varepsilon \text{ for all } n > m > N_1$$

$$\left|\sum_{k=m+1}^{n} z_k\right| < \varepsilon \text{ for all } n > m > N_2$$

Let $N = \max\{N_0, N_1, N_2\}$. Then by assumption,

$$\left|\sum_{k=m+1}^{n} x_k\right| \leq \max\left\{\left|\sum_{k=m+1}^{n} y_k\right|, \left|\sum_{k=m+1}^{n} z_k\right|\right\} < \varepsilon$$

Thus, $\sum x_n$ satisfies the Cauchy criterion, so the series converges.

Study the convergence of $\sum_{n=2}^{\infty} \frac{n^{\log n}}{(\log n)^n}$

Proof: Let $x_n = \frac{n^{\log n}}{(\log n)^n}$. Apply the root test:

$$|x_n|^{\frac{1}{2}} \le \frac{n^{\frac{\log n}{n}}}{\log n} = \frac{e^{\frac{(\log n)^2}{n}}}{\log n}$$

Then, there exists $N \in \mathbb{N}$ s.t. for $n > N$, $\left|\frac{(\log n)^2}{n}\right| \le \frac{\left(n^{\frac{1}{4}}\right)^2}{n} = \frac{1}{n^{\frac{1}{2}}} \le \frac{1}{\sqrt{N}}$, so for $n > N$

$$|x_n|^{\frac{1}{2}} \le \frac{1}{\sqrt{N}} \frac{1}{\log n}$$

Since $\lim_{n\to\infty} \log n = +\infty$ and $\log n \ge 1$ for $n \ge 2$, $\lim_{n\to\infty} \frac{1}{\log n} = 0$. By ALT and squeeze theorem, $\lim_{n\to\infty} |x_n|^{\frac{1}{n}} = 0 < 1$. So by the root test, x_n converges.

Decide if the following series converges: $\sum_{n=1}^{\infty} 2^{-\sqrt{n}}$

Solution: We will use the Cauchy condensation test. Let $x_n = 2^{-\sqrt{n}}$ and consider

$$y_n = 2^n x_{2^n} = e^{n \log 2} e^{-2^{\frac{n}{2}} \log 2} = e^{-(\log 2)(2^{\frac{n}{2}} - n)}.$$

We claim that $2n \le 2^{n/2}$ for all $n \ge 8$. We can prove this by induction on n. The base step is an equality. For the inductive step, we have

$$2(n+1) = 2n + 2 \le 2^{n/2} + 2$$

and we are done provided that we can show that $2^{\frac{n}{2}} + 2 \le 2^{\frac{1}{2}} 2^{\frac{n}{2}} = 2^{\frac{n+1}{2}}$. This is equivalent to

$$2^{\frac{n}{2}} \ge \frac{2}{\sqrt{2} - 1}.$$

We have $n \ge 8$ and the left hand side is increasing so we only need this to be true for $n = 8$ which it as the LHS is 16 whilst the RHS is comfortably less than $\frac{2}{0.25} = 8$. Therefore,

$$y_n \le e^{-c 2^{\frac{n}{2}}} \le e^{-c' n}$$

where $c = (1/2) \log 2$ and $0 < c' < c$. Now this forms a geometric series which converges so the dyadic sum $\sum_{n=0}^{\infty} y_n$ converges and thus by Cauchy condensation, the original series $\sum_{n=1}^{\infty} x_n$ also converges.

Let $A \subseteq \mathbb{R}$ s.t. there exists a sequence $(x_n) \in A$ converging to a real number $x_0 \notin A$. Show there exists an unbounded continuous function on A.

Proof: Let $f : A \to \mathbb{R}$ be given by $f(x) = \frac{1}{x - x_0}$. Since $x_0 \notin A$, $f(x)$ is well-defined on all of A. It is clearly continuous by the ALT. We show it is unbounded. Let $M > 0$ be given and choose $\varepsilon = \frac{1}{M}$. Since $x_n \to x_0$, there exists $N \in \mathbb{N}$ s.t. $|x_n - x_0| < \varepsilon$. So $|f(x)| = \frac{1}{|x_n - x_0|} > M$.

Prove $\lim \frac{n+6}{n^2-6} = 0$

$$\left|\frac{n+6}{n^2-6} - 0\right| < \varepsilon$$
$$\left|\frac{n+6}{n^2-6}\right| < \varepsilon$$

Note that when $n \ge 6$, we have that $|n+6| \le 2n$, $|n^2 - 6| \ge \frac{1}{2}n^2$.

$$\left|\frac{n+6}{n^2-6}\right| \le \frac{2n}{\frac{1}{2}n^2} < \varepsilon$$
$$\frac{4n}{n^2} < \varepsilon$$
$$\frac{4}{n} < \varepsilon$$
$$n > \max\{\frac{4}{\varepsilon}, 6\}$$

Let $\varepsilon > 0$. Let $N \ge \max\{\frac{4}{\varepsilon}, 6\}$. Then $\forall n > N$, we have

$$n > \max\{\frac{4}{\varepsilon}, 6\} \implies \left|\frac{n+6}{n^2-6}\right| \le \frac{2n}{\frac{1}{2}n^2} < \varepsilon$$